Convergence of sequences and series of complex numbers and functions

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Crush course in complex series and sequences:

Nothing new.

Some details: $\{z_n \to z\} \Leftrightarrow \begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z \\ \operatorname{Im} z_n \to \operatorname{Im} z \end{cases}$ $\quad \{z_n \to z\} \Leftrightarrow |z_n - z| \to 0$

$\sum_{n=1}^{\infty} z_n = a \Leftrightarrow \begin{cases} \sum \operatorname{Re} z_n = \operatorname{Re} a \\ \sum \operatorname{Im} z_n = \operatorname{Im} a \end{cases}$

Cauchy test: $\sum z_n$ converges $\Leftrightarrow$ $\forall \varepsilon > 0 \; \exists N: m > n > N: \left| \sum_{k=n}^{m} z_k \right| < \varepsilon.$

Remark $\sum_{n=1}^{\infty} z_n$ - converges $\Rightarrow |z_n| \to 0$ $\quad \not\Leftarrow$

Def. $\sum_{n=1}^{\infty} z_n$ converges absolutely $\Leftrightarrow \sum_{n=1}^{\infty} |z_n| < \infty$

Converging series, important. $\sum_{n=1}^{\infty} z_n$ converges $\Leftrightarrow$ $\sum_{n=k}^{\infty} z_n \to 0$

$A = \left\{ \frac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$ $\quad -1 \quad -\frac{1}{3} \quad \dots \quad \frac{1}{2}$

lower bounds $\qquad$ upper bounds

Supremum and infimum: $A \subset \mathbb{R}$

$a \in \mathbb{R}$ - upper bound for A if $x \in A \Rightarrow x \leq a$
$a = \sup A$ if $\forall a'$ - u.b., $a' \geq a$ - least u.b. $\quad$ no u.b. $\sup A = +\infty$

$a \in \mathbb{R}$ - lower bound for A if $x \in A \Rightarrow x \geq a$
$a = \inf A$ if $\forall a'$ - l.b., $a' \leq a$ - greatest l.b. $\quad$ no l.b. $\inf A = -\infty$

Lim sup and lim inf. (a_n) - a real sequence.

$R_n := \{a_k, k \geq n\}$ $\qquad u_n := \sup R_n \quad u_n \downarrow$
$\qquad \ell_n := \inf R_n \quad \ell_n \uparrow$

$\overline{\lim}$ $\quad \limsup a_n := \lim u_n = \inf u_n$
$\underline{\lim}$ $\quad \liminf a_n := \lim \ell_n = \sup \ell_n$

$\overline{\lim}\, a_n = \sup\{ b : \#\{n : a_n > b\} \text{ is infinite}\} = \inf\{c : \#\{n : a_n > c\} \text{ - finite}\}.$

Example. $a_n = (-1)^n + \frac{\sin n}{n}$ $\quad \overline{\lim}\, a_n = 1, \quad \underline{\lim}\, a_n = -1.$

? $b = 1$ $\quad \#\{n : a_n \geq 1\}$ - infinite. $\quad$ There infinitely many even n: $\sin n > 0$. ?

$b > 1$ $\quad \exists N: \frac{1}{N} < b - 1 \Rightarrow \left(n > N \Rightarrow a_n = (-1)^n + \frac{\sin n}{n} \leq 1 + \frac{1}{n} < b \right.$
$\Rightarrow \left(\{n : a_n \geq b\} \subset \{1, \dots, N\} \right)$ finite.

$b < 1$ $\quad \exists N: \frac{1}{N} < 1 - b \Rightarrow \left(n = 2k, n > N \Rightarrow a_n = 1 - \frac{\sin n}{n} \geq 1 - \frac{1}{n} > b \right)$

$\#\{n : a_n > b\}$ - infinite.

Property. $u = \overline{\lim}\, a_n$ is the unique number such that: 1) $\forall \varepsilon > 0: \{n : a_n > u + \varepsilon\}$ is finite
2) $\forall \varepsilon > 0: \{n : a_n > u - \varepsilon\}$ - infinite.

Another characterization: $\overline{\lim}\, a_n = \sup\{b : \#\{n : a_n > b\}$ - infinite$\} =$
$\inf\{b : \#\{n : a_n > b\}$ - finite$\}$.

$\underline{\lim}\, a_n = \inf\{b : \#\{n : a_n < b\}$ - infinite$\} =$
$\sup\{b : \#\{n : a_n < b\}$ - finite$\}$

$\lim_{n \to \infty} a_n$ - exists if and only if $\overline{\lim}_{n \to \infty} a_n = \underline{\lim}_{n \to \infty} a_n.$

Bonus problem (+1 pt):

Let $a_n > 0$ be an unbounded sequence.
Prove that $\overline{\lim} \left(1 + \frac{1}{a_n}\right)^{a_n} = e.$

$\overline{\lim}\, a_n \log\left(1 + \frac{1}{a_n}\right) = 1.$

Uniform vs pointwise $\quad f_n, f : K \to \mathbb{C}, \quad K \subset \mathbb{C}.$

$f_n \to f$ pointwise if $\forall z \in K\ \forall \varepsilon > 0 \; \exists N(z, \varepsilon) : n > N(z, \varepsilon)\ |f_n(z) - f(z)| < \varepsilon \Leftrightarrow \forall z \in K\ |f_n(z) - f(z)| \to 0$

$f_n \rightrightarrows f$ - uniformly if $\forall \varepsilon > 0 \; \exists N(\varepsilon) . n > N \ |f_n(z) - f(z)| < \varepsilon \Leftrightarrow \sup |f_n(z) - f(z)| \to 0$

$z \in k$

Thm. Uniform limit of a sequence of continuous functions is continuous. $\exists \varepsilon_n \to 0: \forall z \in k \ |f_n(z) - f(z)| < \varepsilon_n$

Same proof as in real case.

Example

On $[0,1]$, $x^n \to \begin{cases} 0, & x<1 \\ 1, & x=1 \end{cases}$

M-test for uniform convergence of series.

Thm. $k \subset \mathbb{C}$. $\sum_{n=1}^{\infty} M_n < \infty$ $\quad M_n \geq 0$.

Then if $\forall z \in k$, $\forall n$: $|f_n(z)| \leq M_n$, then $\sum f_n(z)$ converges uniformly on k.

Proof. $\sum M_n$ - converges $\Rightarrow$ $\forall \varepsilon > 0$ $\exists N: m > n > N$ $\sum_{k=n}^{m} M_k < \varepsilon \Rightarrow$

$\forall \varepsilon > 0$ $\exists N: m > n > N$ $\sum_{k=n}^{m} |f_k(z)| < \varepsilon \Rightarrow \sum_{k=1}^{\infty} f_k(z) =: f(z)$ exists $\forall z \in k$.

$|f(z) - \sum_{k=1}^{n} f_k(z)| < \sum_{k=n+1}^{\infty} M_k \to 0$, so uniform ∎